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MTS.
Rigidity in alg. K-theory and TC.

Joint w/ Clausen and Morrow.

R comm ring.

$K(R)$ connective K-theory.

Ex. $K_n(\mathbb{Z}) = 0?$, $n \geq 1$.

Def. $I \subseteq R$ an ideal.

$$K(R,I) \to K(R) \to K(R/I).$$

Maybe $K(R,I)$ is easier to understand.

Ex. R a $\mathbb{Z}[1/\ell]$-algebra, I nilpotent.

Then, $K(R,I)/\!/\ell \simeq 0$.

Can see this using $H_*(GL_\infty(R), \mathbb{Z}/\ell) \cong H_*(GL_\infty(R/I), \mathbb{Z}/\ell)$ using Serre s.s.

Ex. $K(\mathbb{Z}_p)^\wedge_\ell \simeq K(\mathbb{F}_p)^\wedge_\ell$.

Def. (R,I) a pair, $I \subseteq R$ an ideal.

This is henselian if $f(x) \in R[x]$ and $\bar{\alpha} \in R/I$ s.t. $F(\bar{\alpha}) = 0$, and $f'(\bar{\alpha})$ is a unit, then there is α in R (lifting $\bar{\alpha}$) s.t. $f(\alpha) = 0$.

Rem. In fact, α is unique and ~~[illegible]~~ $I \subseteq \mathrm{rad}(R)$.

Ex. R I-adically complete, then (R,I) is henselian.
Hensel's Lemma.

Ex. R germs of holomorphic functions in a nbd of $0 \in \mathbb{C}$.
Contained in $\mathbb{C}[[z]]$.

Ex. $R \subseteq \mathbb{C}[[z]]$ satisfying an alg. equation over $\mathbb{C}[z]$.
Ex, $\sqrt{1+z}$. $R = \mathbb{C}[z]^h_{(0)}$.

Ex. I nilpotent $\Rightarrow$ I henselian.

Thm (Gabber, Gillet-Thomason, Suslin).

1) If (R,I) is henselian, ℓ invertible on R, then $K(R,I)^\wedge_\ell \simeq 0$.

True in nonconnective K-theory?

2) If (R,m) is a hensel local ring (m the max. ideal), ℓ invertible, R/m sep. closed, then
$$K(R)^\wedge_\ell \simeq ku^\wedge_\ell.$$

I guess not: there are hensel local rings with $K_{-2} \neq 0$. Maybe it disappears on completions?

Goal: p-adic results for p-adic rings.
Write $\hat{K}(R) = K(R)^\wedge_p$.

Exs. $\hat{K}(\mathbb{F}_p) \simeq \begin{cases} \mathbb{Z}_p & * = 0, \\ 0 & \text{otherwise.} \end{cases}$

$$\hat{K}(\mathbb{Z}/p^2) \simeq \begin{cases} \mathbb{Z}_p & * = 0, \\ \text{unknown} & p\text{-torsion.} \end{cases}$$

TC. Interested in p-adic version.

Def. p-complete. $X \in \mathrm{Sp}_p^{BS^1}$.

$$X \xrightarrow{\varphi} X^{tC_p}.$$

S^1-equivariant.

- $THH(R)$, a cyclotomic spectrum.
- $TC(R) \simeq \underline{\mathrm{Map}}_{\mathrm{CycSp}}(\mathbb{1}, THH(R))$.
- $TC(R) \xrightarrow{\otimes} TC^-(R) \underset{\varphi^{hS^1}}{\overset{\mathrm{can}}{\rightrightarrows}} TP(R)$.
- $K(R) \xrightarrow[\text{trace}]{\text{cyclotomic}} TC(R)$, often a good approximation for p-adic rings.
- **Ex.** $\pi_* TC(\mathbb{F}_p) \simeq \begin{cases} \mathbb{Z}_p & * = 0, -1, \\ 0 & \text{otherwise.} \end{cases}$

Thm (Hesselholt–Madsen). R f.g. commutative algebra over $W(k)$, k perfect of char p, then

$$\hat{K}(R) \xrightarrow{\sim} \tau_{\geq 0} \hat{TC}(R).$$

Application. $\hat{K}(\mathcal{O}_K)$, $K/\mathbb{Q}_p$ finite. Verifies Bloch–Kato.

Thm (Dundas-Goodwillie-McCarthy). (R,I) a pair, I nilpotent.

$$K(R,I) \simeq TC(R,I).$$

Also due to McCarthy in the p-complete case.

Thm (CMM). If (R,I) is a henselian pair, then

$$K(R,I)/p \simeq TC(R,I)/p.$$

Remk. Generalizes GGT rigidity and DGM.

Cor. If (R,m) is a henselian local ring, R/m sep. closed of char. p, then

$$K(R)/p \simeq TC(R)/p.$$

Rems. (1) Use Geisser-Levine's results that compute

$$K\left(\begin{matrix}\text{reg. local}\\ \mathbb{F}_p\text{-algebras}\end{matrix}\right)/p.$$

Also can use Geisser-Hesselholt computations for TC/p. Known and in terms of differential forms. This handles the char. p case.

(2) TC/p commutes with filtered colimits as $\mathrm{CAlg}_{\mathbb{Z}} \longrightarrow \mathrm{Spectra}$. Failure for TC^- and TP cancels out. True in fact on $\mathrm{CycSp}_{\geq 0}$.

Henselian condition only depends on the ideal I as a non-unital objects.
Use free objects. Then, colim over. xn. Get something natural.

(3) Gabber. ~~[illegible]~~ Reduce to henselian pairs over fields.
Need to use some approximations and hence (2).
Use McCarthy's thm too. Also rigidity.

<u>Applications.</u>

For nice p-adic rings, p-adic K-theory is asymptotically TC.

A) <u>Thm</u> (CMM). R p-complete, R/p has finite Krull dim,
$d = \max\left(1, \sup_{\mathfrak{p} \in \operatorname{Spec} R/p} [k(\mathfrak{p}) : k(\mathfrak{p})^p]\right)$. Then,

$$\tau_{\geq d} \hat{K}(R) \simeq \tau_{\geq d} \widehat{TC}(R).$$

B) Continuity in K-theory. R I-complete. How close is

$$K(R)/p \longrightarrow \lim K(R/I^n)/p$$

to an $\simeq$?

<u>Ex</u>. If p is invertible, both sides are the same by rigidity; tower is constant.

Previous results mostly for $\mathbb{F}_p$-algebras.

<u>Thm</u> (CMM). R noetherian, I-complete, R/p is F-finite.
Then, $K(R)/p \xrightarrow{\sim} \lim K(R/I^n)/p$.

<u>Rem</u>. Continuity can fail w/o F-finiteness for rings like $k[[t]]$, $[k:k^p] = \infty$.

<u>Rem</u>. Actually $\simeq$ to the henselian pair result by some comm. alg.

<u>Rem</u>. Can reduce to $TC(R)/p \simeq \lim TC(R/I^n)/p$, due to Dundas-McCarthy.

<u>Ex</u>. $\Omega^1_{\mathbb{F}_p[[t]]/\mathbb{F}_p} \simeq \mathbb{F}_p[[t]] \cdot dt$.

Char. p Magic.